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EFFECT OF STABILIZING FORCES ON TURBULENCE

By L. Prandtl

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By L. Prandtl

I have frequently noticed on pleasant summer evenings that a temperature layer, such as is caused by the evening radiation and the ensuing cooling off of the air strata near to the earth's surface, manifests itself in the sudden cessation of the turbulence of the wind and flows on without any sign of forming clouds. There is but one interpretation for this, namely, the warm air glides in laminas over the cold layers of air beneath it. The appearance of expressed discontinuity layers in the free atmosphere, where a warm air mass flows over a cold mass without perceptible intermingling, belongs in the same group, so also the flow of fresh water over sea water as is particularly observed in the polar ocean. The opposite phenomenon is that of increased turbulence due to convective motions ensuing from weight differences when the earth and the bordering lowest layers of air or water are warmed up. In the first case the "exchange" as well as the friction of the flowing air or water mass is lowered, in the second case it is raised. Since these interrogatory forms were in close relation to our Göttingen test program on fully developed turbulence,** I acquiesced to the request of Wilhelm Schmidt of Vienna to undertake a study of this subject. In these experiments an air stream is to be blown between a water-cooled and a vapor-heated plate. According to what preceded it is anticipated that the exchange will be lowered appreciably or even stopped altogether under appropriate conditions if the stratum is staple (warm above, cold below), while in the contrary case the exchange will be enhanced. Because these experiments had to be made at relatively low air speeds so as to let the lifting forces become effective, it first required a study of the test methods, and so we have not yet proceeded beyond the preliminary tests on the most suitable hot-wire connections and thermocouples for recording speed and temperature. The wish to predict the phenomena which we want to observe here prompted me, however, to search into the theory of these phenomena.

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*"Einfluss stabilisierender Kräfte auf die Turbulenz," from a reprint of Vorträge aus dem Gebiete der Aerodynamik und verwandter Gebiete, Aachen, 1929, issued by A. Gilles, L. Hopf and Th. v. Kármán.

**N.A.C.A. Technical Memorandum No. 435, Sept., 1926, "Turbulent Flow."

In addition I was hopeful that this aspect would be productive of enhanced insight into the mechanism of turbulence.

The momentary status of this theory is as follows: Assume a horizontal fluid flow of vertical laminas so that each superposed stratum is lighter than the one below it. Then, when the motion is turbulent it produces energy, e.g., the heavier layer is raised and the lighter layer is lowered against the lift of the heavier. Let the path in the vertical, traversed by a fluid portion prior to intermingling anew with its new surrounding be designated by l. This is the so-called "mixing path or distance." If $-d\rho/dy$ expresses the decrease in density in the vertical (y-direction), then the ultimate lift difference of a particle shifted upward becomes $= -gl\frac{d\rho}{dy}$. Since this lift difference increases from zero to this value while covering distance l, we have an energy output of $-\frac{1}{2}gl^2\frac{d\rho}{dy}$ (positive when, as assumed here, $d\rho/dy$ is negative!). Downward motions yield the same expression. In order to embrace the volumes participating on the exchange, we draw a horizontal surface and attribute to one fraction β_1 thereof an upward motion with speed v_1, and to another fraction β_2 a downward motion with mean speed v_2. Accordingly the total amount on surface F is $F(\beta_1 v_1 + \beta_2 v_2)$ (volume per second), thus making available as performance the product of the "apparent shearing stresses" called forth by the exchange and the rate of slippage. When we bring this performance into relation with a body of base F and height l (Fig. 1),

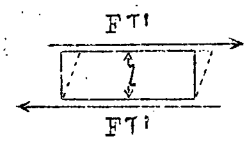

Fig.1

and when τ' is the apparent shear stress and $\bar{u}$ the mean rate of the flow, this performance becomes $= F\tau' l \frac{d\bar{u}}{dy}$. The selection of a layer of height l is justified, for the previously conceded fluid quantity is precisely transmitted in such a layer. Conformal to O. Reynolds the apparent shear stress is $= \overline{\rho u'v'}$, the $\overline{\text{line}}$ denoting a formed average. Accordingly the mean of

$u' = l\frac{d\bar{u}}{dy}$ and of v' is $= \beta_1 v_1 + \beta_2 v_2$. Since the sign for v' fluctuates regularly with that of u' we write

$$\tau' = \rho\, l \frac{d\bar{u}}{dy} (\beta_1 v_1 + \beta_2 v_2).$$

The existence or nonexistence of energy for the further maintenance of the flow in question now depends on which of the two energy figures predominates. If the energy output exceeds the shear stress, the difference is available for upholding the turbulence, but if the lift performance is greater than the weight difference, the turbulence must perforce die away. Thus something is left to maintain the turbulence when

$$F \tau' l \frac{d\bar{u}}{dy} > F(\beta_1 v_1 + \beta_2 v_2)\ \tfrac{1}{2} g\, l^2 \frac{d\rho}{dy}, \quad \text{or,}$$

after inserting the value of τ' and abbreviating

$$\rho\left(\frac{d\bar{u}}{dy}\right)^2 > -\frac{g}{2}\frac{d\rho}{dy}. \tag{1}$$

One notable feature is the elimination of the chosen surface F, the mixing distance l and the mean value of vertical speed v' from our inequation (1), so that this interpretation does not reveal anything about these two important quantities. In addition, it is advisable to designate the quotient at the right and left sides of (1) separately. Making this nondimensional quantity $= \theta$:

$$\theta = \frac{-\,g\frac{d\rho}{dy}}{2\rho\left(\frac{d\bar{u}}{dy}\right)^2}, \tag{2}$$

yields turbulence for $\theta < 1$ and strong stability; the turbulence dies out when $\theta > 1$. $\theta = 0$ denotes that the fluid is homogeneous. Since the derivation of the preceding formulas is primarily a rough estimate, the experiments undoubtedly call for some kind of correction factor. But from the construction of the formulas it may be inferred that, for instance, what the relevant quantities are and what qualitative course may be expected. An inherently homogeneous warm air stream flows over an identically cold one; for instance, $d\bar{u}/dy$ may temporarily be very high in the transition layer between the two, so as to comply with inequation (1), that is, prevailing turbulence. During the turbulent exchange the mixing distance becomes gradually thicker, and $d\bar{u}/dy$ conformably smaller. This also applies to the transition of the temperature. Now the subsequent limiting attitude is plausible where inequation (1) becomes precisely the equation and then protracts the attained attitude because of the discontinued turbulence (the pure viscosity effects and the thermal conductivity are to be small enough to become insignificant). Now, with h as the height of the transition layer in which linear speed distribution prevails, we write $\frac{d\bar{u}}{dy} = \frac{u_2 - u_1}{h}$, and instead of $-\frac{1}{\rho}\frac{d\rho}{dy}$ with T = absolute temperature, we put $\frac{1}{T}\frac{dT}{dy} = \frac{1}{T_m}\frac{T_2 - T_1}{h}$.*

This yields

$$\left(\frac{u_2 - u_1}{h}\right)^2 = \frac{g}{2T_m}\frac{T_2 - T_1}{h}$$

or

$$h = \frac{2(u_2 - u_1)^2 T_m}{g(T_2 - T_1)} . \tag{3}$$

This equation (3) gives the minimum value of the height of such a transition layer at which the flow is without turbulence. The comparison with empirical figures should show the accuracy of the numerical factor of this formula or any eventual correction, although a precursory collation attested to the conformity of (3) with experience.

The shear stress τ' of the apparent friction may be expressed by

$$\tau' = \rho l^2 \left(\frac{du}{dy}\right)^2 f(\theta), \tag{4}$$

regarding which we, for the time being, merely say that $f(\theta)$ must become $= 0$ when $\theta \geqq 1$, and $f(\theta)$ must become $= 1$ when $\theta = 0$ (ordinary turbulence in homogeneous fluids). On the premises of an energy consideration (intensity of turbulent motion proportional to the difference of the above explained energies), it is concluded that $f(\theta) = \sqrt{1 - \theta}$, subject how-

*The difference of the potential temperatures should, strictly speaking, be used here for this temperature difference so as to embrace the temperature changes due to changes in heights as the pressure varies.

ever to further proof. Above all it is anticipated that mixing distance l will be affected by θ, a thought which might be extended to the case of initially unstable stratification, or in other words, that the air below is warmer than above. Such conditions may arise for longer periods when the air is warmed up from the ground. In this case the mixing distances will presumably be greater due to the brisk thermal convection than in the case of ordinary turbulence.

Closely bound up with these conditions of laminar flow are the curved flows of a homogeneous fluid, in our particular case the plane flow in concentric circles. Let the mean speed in tangential direction $\bar{u}$ be a function of the radius. When the speed decreases from the outside toward the inside the fluid particles from the inner zone can no longer penetrate the outer flow, for their centrifugal force is less than that of the outer particles, and conversely can no longer hold the particles which came from without, but return whence they came. Conversely, instability prevails when the inward speed rises sufficiently. To each particle motivated frictionless in the remaining medium the surface theorem expressed as u = tangential component of the speed $u\,r$ = constant is applicable, because the pressure field is precisely symmetrical in rotation. As soon as a particle is radially shifted for an amount dr, its speed obviously changes by $-\frac{u}{r}dr$. If the mean speed at the new position differs from the old by $\frac{d\bar{u}}{dr}dr$, a radial slippage for an amount l yields in first approximation a speed difference

$$u' = l\left(\frac{d\bar{u}}{dr} + \frac{\bar{u}}{r}\right). \tag{5}$$

This difference becomes zero for the potential flow with circulation, $\bar{u} = \frac{\text{constant}}{r}$, and consequently plays the same role with respect to turbulent motion as the motion with constant speed in the rectilinear flow. A particle which differs in speed u' from its surrounding shows a difference in centrifugal force per unit volume with respect to that in the vicinity $\frac{\rho(\bar{u} + u')^2 - \bar{u}^2}{r}$, that is, $\frac{2\rho\bar{u}u'}{r}$ in first approximation.

Thus we find in connection with formula (5) that potential motion $\bar{u}\,r$ = constant is an exact parallel of the case of neutral equilibrium. An outward increase of product $\bar{u}\,r$ denotes a stable arrangement, an outward decrease, an unstable process. In the latter case we encounter the vortex motions of which G. I. Taylor had made a particular study.*

Now we revert to the two energy outputs. As far as the energy against the centrifugal force is concerned this is for a slippage of $l = l/2$ times the defined difference in centrifugal force per unit of volume, or

$$\frac{\rho l \bar{u} u'}{r} = \rho l^2 \frac{\bar{u}}{r} \left(\frac{d\bar{u}}{dr} + \frac{\bar{u}}{r}\right).$$

The amount of the additive fluid is computed as before. When defining the adduced output it must be remembered that the speed of the relative slippage of two intersecting surfaces (speed of deformation) distant by the length unit is here $\frac{d\bar{u}}{dr} - \frac{\bar{u}}{r}$ (equal zero for the rotation as rigid body $\bar{u} = \omega r$). Then the output becomes $F \tau' l \left(\frac{d\bar{u}}{dr} - \frac{\bar{u}}{r}\right)$, where again

$$\tau' = \rho \overline{u'v'} = \rho l \left(\frac{d\bar{u}}{dr} + \frac{\bar{u}}{r}\right) (\beta_1 v_1 + \beta_2 v_2),$$

and the formula expressing the possibility of turbulence finally becomes

$$\left(\frac{d\bar{u}}{dr} + \frac{\bar{u}}{r}\right) \left(\frac{d\bar{u}}{dr} - \frac{\bar{u}}{r}\right) > \left(\frac{d\bar{u}}{dr} + \frac{\bar{u}}{r}\right) \frac{\bar{u}}{r} . \tag{6}$$

If $\frac{d\bar{u}}{dr} + \frac{\bar{u}}{r}$ is positive, it yields $\frac{d\bar{u}}{dr} > 2\frac{\bar{u}}{r}$, if $\frac{d\bar{u}}{dr} + \frac{\bar{u}}{r}$ = negative, we have $\frac{d\bar{u}}{dr} < 2\frac{\bar{u}}{r}$. But the latter inequation, positive $\bar{u}$ assumed, is always complied with for $\frac{d\bar{u}}{dr} < -\frac{\bar{u}}{r}$ according to assumption.

The conditions are more conveniently perceived when graphing the right and left sides of inequation (6), first dividing by $\left(\frac{\bar{u}}{r}\right)^2$ and using $\frac{d\bar{u}}{dr} : \frac{\bar{u}}{r}$ as abscissa, as transscribed on Figure 2, where the zones of anticipated turbulence, i.e., where (6) is fulfilled, are shown as shaded portions. An examination of temporary typical flows whose speed distribution conforms to power law $\bar{u} = ar^n$ yields:

*Taylor, G. I.: "Stability of a Viscous Liquid Contained between two Rotating Cylinders," Phil. Trans. Roy. Soc., Vol. 223 A, 1923, p. 289.

instability, Taylor vortices	for	$n < -1$
indifference, potential flow	"	$n = -1$
stability, laminar flow	"	$-1 < n < +2$
stability, rigid rotation	"	$n = +1$
stability, turbulent flow	"	$n > +2$.

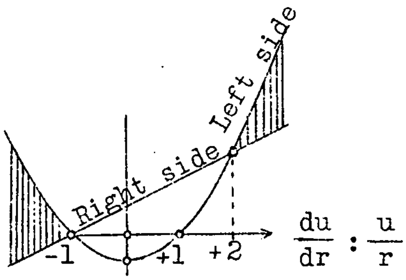

Fig.2

Naturally, it must be proved by experiment whether the limit for the turbulence in the outward increasing speeds lies actually at $n = 2$ or at some other figure. However, precursory studies revealed that the limit value is not far from 2.

In order to obtain a formula for the shear stress we again have recourse to the ratio of the right side of inequation (6) to the left side as parameter, so that when making it = θ_1, we can write

$$\tau' = \rho l^2 \left(\frac{d\bar{u}}{dr} + \frac{\bar{u}}{r}\right)^2 f(\theta_1).$$

Then $f(\theta_1) = 0$ for $\theta_1 \geqq 1$ and $= \sqrt{1 - \theta_1}$ for smaller values.* The latter reference is yet without tangible information so long as we do not know how the mixing length l is dependent on θ_1. It follows that the aim should be to so define function $f(\theta_1)$ that l would be contingent on the geometrical configuration of the cylinder but not on the speed distribution. The possible ambit of this hinges on the studies under preparation to this end.

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* It becomes readily apparent that $\theta_1 = 0$ is conformal to the rectilinear motion $(r = \infty)$, so that our formula here reduces again to the usual form.

As regards these experiments they pertain, in a few words, to water between two concentric cylinders, each of which can be driven at a different rate of rotation. The torque transmitted through the water is to be measured and the speed and pressure distribution defined.

Commentary by J. M. Burgers, Delft

The effect of a density layer on the stability of flow was studied by L. F. Richardson.* His deductions relate to the problem of atmospheric flow with thermodynamic relations and he adduces the decay of the turbulence when

$$\frac{g}{\theta}\left(\frac{d\theta}{dh} + \frac{g}{c_p}\right) > \left(\frac{\partial \bar{v}_x}{\partial h}\right)^2 + \left(\frac{\partial \bar{v}_y}{\partial h}\right)^2 .$$

Here θ = absolute temperature, g = acceleration of gravity, c_p = specific heat of air at constant pressure, h = height, $\bar{v}_x$ and $\bar{v}_y$ = the horizontal components of the mean speed. With $\theta_p = \theta + \frac{gh}{c_p}$, θ_p becomes the potential temperature and the first term of the above inequation assumes the form $\frac{g}{\theta}\frac{d\theta_p}{dh}$. A series of observations made by Benson revealed a satisfactory affirmation of the formula. It is manifest that this criterion agrees with Prandtl's reference. Richardson's studies may be applied direct to a simple case where the thermal expansion is without any significance.

I may be allowed to add a few remarks relative to the formula for the exchange coefficient, particularly when the mean flow has only one speed component u in direction x which is a function of y, all other quantities in question to be functions of y. According to definition we then obtain the value of the exchange coefficient in the neighborhood of a point P by laying a surface S through P parallel to the x-y plane and defining at moment t for each point of this surface the momentary value of v' (i.e., the speed component in direction y) and of distance l' traversed by the volume element passing through the relevant point since the "last compensation process" in positive direction y. These distances l' may take a different turn according to whether

*L. F. Richardson: Proc. Roy. Soc. London, Vol. 97, 1920, p. 354. Phil. Mag. Vol. 49, 1925, p. 81.

we stress the compensation of the mean speed, or temperature, or of the concentration of a detached substance, etc. Each special case then yields the pertinent exchange coefficient as the mean value

$$\epsilon = \overline{v' l'}$$

over surface S.

Now the expression "last compensation" is not easily defined, but we may, if we first study the compensation of the speed, arrive at a satisfactory formula as indicated by G. I. Taylor,* when we substitute for l' the distances l which the relevant volume elements have traversed since a stated time interval $T(<t)$ in the positive direction y. Hereby interval $t - T$ must be chosen large enough to ensure at least one speed compensation in each volume element. Whereas v' and l' (i.e., the speed in direction y at moment t and the distance traversed since the last compensation) always have the same sign, there must, if the conception "last compensation" is correctly interpreted, exist no correlation between v' and $l - l'$. Consequently

$$\overline{v' l'} = \overline{v' l} - \overline{v'(l - l')} = \overline{v' l}.$$

The last quantity is unobjectionably defined and may, according to Taylor, be used in the formulas of the speed exchange.

Taylor, moreover, related quantity $\overline{v' l}$ with two other quantities. It is patent that

$$\overline{v' l} = \overline{\frac{dl}{dt} l} = \frac{1}{2} \overline{\frac{d}{dt} l^2} = \frac{1}{2} \frac{d}{dt} \overline{(l^2)},$$

(because forming the mean value always pertains to the attitude at moment t on the fixed surface S, thus permitting the change in consecutive order of differentiation and mean values).

On the other hand

$$\overline{v' l} = \overline{\left(v'_t \int_T^t v'_\tau d\tau\right)} = \int_T^t \overline{v'_t v'_\tau} d\tau,$$

*Taylor, G.I.: Proc. London Math. Soc., Vol. 2, 1922, p. 196.

when v'_τ is the speed in direction y at interval τ, of the same volume element to which v'_t is relevant; (the formation of the mean value is always with respect to surface S in time interval t).

With

$$\overline{v'_t v'_\tau} = \overline{v'^2} R_{t-\tau},$$

we have

$$\overline{v' l} = \overline{v'^2} \int_\tau^t R_{t-\tau} d\tau = \overline{v'^2} \int_0^{t-\tau} R_\xi d\xi.$$

R_ξ is a so-called correlation coefficient, where the fact that R_ξ is a linear function of ξ, has been utilized. It naturally is now assumed that $\int_0^\infty R_\xi d\xi$ has a finite value; it is really this limiting value which defines the value of $\overline{v' l}$. Consequently

$$\epsilon = \frac{1}{2} \frac{d}{dt}\overline{(l^2)} = \overline{v' l} = \overline{v'^2} \int_0^\infty R_\xi d\xi.$$

Quantity $\overline{l^2}$ is related with a similar one occurring in the theory of Brown's motion. On the other hand, mean value $\overline{v'_t v'_\tau}$ belongs to the type of "correlation moments" introduced by Keller and Friedmann in their general theory on turbulence.*

Heretofore we were concerned with the exchange of speed or momentum. Now we turn to the exchange of an arbitrary quantity and express the relevant exchange coefficient by

$$\epsilon_i = v' l'_i.$$

Now we must distinguish two cases: the exchange of the considered quantities is slower than that of the speed or that of the relevant quantities is faster than that of the speed. In the former, l'_i is generally measured from an earlier time interval than l', thus precluding any correlation between v' and $l'_i - l'$. Then

*Keller, L. and A. Friedmann: Proc. Ist. Intern. Congress Applied Mechanics, Delft, 1924, p. 395.

$$\epsilon_i = \overline{v'l'} + \overline{v'(l'_i - l')} = \overline{v'l'} = \epsilon$$

In the latter case, l'_i is measured from a later time interval than l', so that $l' - l'_i$ usually has the same prefix as v', and we have:

$$\epsilon_i = \overline{v'l'} - \overline{v'(l' - l'_i)} < \overline{v'l'} \text{ or } < \epsilon.$$

The derivation of the Richardson formula is bound up with the exchange coefficient for density gradients. The volume elements passing in interval t through S have the density

$$\rho_P - l'_i \frac{d\rho}{dh},$$

and are, in consequence, pulled downward with a force

$$- gl'_i \frac{d\rho}{dh} .$$

The force acting against the gravity in unit time per unit of volume is

$$- gv'l'_i \frac{d\rho}{dh} .$$

So when we presume the density gradients to compensate at least not faster than those of the speed, we can substitute ϵ for $\overline{v'l'_i}$.

With respect to the last point it should be borne in mind that the compensation of differences in density by pure diffusion is usually slower than that of speed differences by internal friction. Because $\lambda/\mu c_p$ is greater than 1 for many gases, the compensation of temperature differences may proceed more rapidly than that of the speed differences by internal friction alone; but, since the latter compensation is aided by the pressure effects no danger is likely to occur.

Translation by J. Vanier,
National Advisory Committee
for Aeronautics.